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A new kind of deformed calculus and parabosonic coordinate representation

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Abstract. The general expression for the momentum operator P of parabose quantization theory in the coordinate x-diagonal representation is derived from the Heisenberg equations of motion. Based on this a new kind of deformed calculus is developed. By its application, the eigenequation for a free parabosonic Hamiltonian in coordinate representation is solved and the results correspond exactly to the well known ones in number representation.

1. Introduction

In the last few years there has been increasing interest in generalized statistics. The main reason is their possible application to the theory of fractional quantum Hall effect [1] and to anyon superconductivity [2]. Haldane fractional statistics [3], generalizing the Pauli exclusion principle to any spatial dimension, has also attracted much attention. A large class of generalizations is based on permutation group invariance, for example, parastatistics [4] is the statistics of identical particles defined by the irreducible representations of the permutation group characterized by the triangular Young tables. This parastatistics, carried out at the level of the algebra of creation and annihilation operators, involves trilinear commutation relations in place of the bilinear relations that characterize Bose and Fermi systems. Even though there are no observed paraparticles in nature, the possibility exists for unobserved particles which obey the parastatistics. Some developments in interacting many-particle systems have also shown that the quasiparticles in such systems may exhibit features far more exotic than those permitted to ordinary particles [5], and it appears quite possible that parastatistics may be realized in condensed matter physics.

In seeking such applications of parastatistics, it is essential that one has a complete knowledge of coordinate or momentum representation, besides number representation, for ideal parasystems, because in ordinary quantum mechanics the coordinate or the momentum representations are of central importance. On the other hand, deformations of standard mathematical objects have also attracted a lot of attention recently [6]. The emergence of deformed algebraic structures in the studies of several physical theories has aroused interest leading to an extensive exploration of possible deformations of several well known models of physical phenomena [7]. In particular, the quantum group representation theory and the non-commutative space have been studied extensively and a covariant differential calculus on the quantum hyperplane has been presented [8]. A few years ago, the canonical partition

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function for a non-trivial parasystem, a parasystem with order two, was derived [9], and the corresponding results for any order were obtained only two years ago [10].

The aim of this paper is to construct the coordinate representation for a paraboson system. The crucial point of our approach is deriving the most general expression for the momentum operator P in the coordinate x-diagonal representation according to the paraquantization principle. In section 2 we deduce such an expression for the momentum P starting from the Heisenberg equations of motion, which can be considered as a deformation of ordinary momentum operators in coordinate representation. Based on the most general expression of P, a new kind of deformed calculus is developed in section 3. We solve the eigenequation of free parabosonic Hamiltonian and obtain a whole spectrum of eigenvalues and eigenfunctions in section 4. Using the deformed calculus developed in section 3, the normalization constants of these eigenfunctions are easily derived. The relation to the parabose number representation and some conclusions are also discussed in the last section.

2. The coordinate and the momentum operators

Green generalized the usual quantum statistics by postulating double commutation relations among fields as alternative solutions of the Heisenberg equations of motion [4]. Taking parabose relations in this discussion, the Heisenberg equations of motion are of the form

$$\dot{P} = \mathrm{i}[H, P] = -x \qquad \dot{x} = \mathrm{i}[H, x] = P \tag{1}$$

where $[,]$ is a commutator of operators, x and P are the coordinate and the momentum operators respectively, and $H = 1/2(P^2 + x^2)$. For the sake of simplicity, we have taken x, P and H as dimensionless quantities and only dealt with a single parabose degree of freedom case. In the following discussion, we take equation (1) as our basic commutation relation.

Introducing a notation $S = [x, P] - \mathrm{i}$, from equation (1) we see that

$$\{S, x\} = 0 \qquad \{S, P\} = 0 \tag{2}$$

where $\{,\}$ is an anticommutator of operators. Equation (2) implies that S^2 commute with both the operators x and P. Thus, S^2 must be a c-number. Obviously, $S = 0$ is the normal statistics case. Comparing with the well known Fock space structure for the parabosons [11], we can find in fact $S^2 = -(p-1)^2$, where p is the so-called order of paraquantization and may take any non-negative real number. We can ask what is the most general expression for the momentum operator P in the coordinate x-diagonal representation? By the x-diagonal representation, we mean

$$x|x'\rangle = x'|x'\rangle \qquad \langle x|x'\rangle = \delta(x - x'). \tag{3}$$

In this x-diagonal representation, $\{S, x\} = 0$ leads to an equation

$$0 = \langle x'|\{S, x\}|x''\rangle = (x' + x'')\langle x'|S|x''\rangle \tag{4}$$

which has an obvious solution, i.e.

$$\langle x'|S|x''\rangle = \mathrm{i}c(x')\delta(x' + x'') \tag{5}$$

where the function $c(x')$ satisfies $c^*(x') = c(-x')$, because S is an anti-Hermite operator.

We define a coordinate reflection operator R in the coordinate representation, such that

$$R|x'\rangle = |-x'\rangle \tag{6}$$

which implies that $\langle x'|R|x''\rangle = \delta(x' + x'')$. Comparing this relation with equation (5) and noticing that $\langle x'|$ and $|x''\rangle$ are arbitrary bra and ket eigenvectors of the coordinate operator x in the coordinate representation, we can write S as

$$S = \mathrm{i}c(x)R \tag{7}$$

which implies that $\{R, x\} = 0$. Now we can present the most general expression for the momentum operator P in the coordinate representation as

$$P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\frac{c(x)}{2x}(1 - R) \tag{8}$$

which will satisfy all the requirements on P. For example, by substituting equation (8) into the commutator $[x, P]$ gives $\mathrm{i} + S$, as it should be. Then, by substituting equation (8) into the anticommutator $\{S, P\} = 0$ will lead to the following equation (noticing $R(\mathrm{d}/\mathrm{d}x) = -(\mathrm{d}/\mathrm{d}x)R$)

$$\frac{\mathrm{d}c(x)}{\mathrm{d}x} + \frac{c^2(x)}{2x} - \frac{c(x)c(-x)}{2x} = 0 \tag{9}$$

which has a simple solution, i.e. $c(x) = c$, a real number. Since we know that $S^2 = -(p-1)^2$, we can simply take $c = p - 1$ in our following discussion. Therefore, the momentum operator P for a parabosonic system is of [12]†

$$P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\frac{p-1}{2x}(1 - R) \tag{10}$$

which anticommutes with R. Obviously, when $p = 1$, it reduces to the ordinary momentum operator in coordinate representation.

3. A new kind of deformed calculus

From equation (10) we can define a new derivative operator D which acts on functions $f(x)$ of the real variable x as

$$Df(x) \equiv \frac{D}{Dx}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{p-1}{2x}(1-R)f(x) = \mathrm{d}f(x) + \frac{p-1}{2x}(f(x) - f(-x)) \tag{11}$$

where $\mathrm{d}f = (\mathrm{d}/\mathrm{d}x)f$. Equation (11) means D acts on an even function $f_\mathrm{e}(-x) = f_\mathrm{e}(x)$ as the ordinary derivative $Df_\mathrm{e}(x) = \mathrm{d}f_\mathrm{e}(x)$, and D acting on an odd function $f_\mathrm{o}(-x) = -f_\mathrm{o}(x)$ leads to $Df_\mathrm{o}(x) = \mathrm{d}f_\mathrm{o}(x) + ((p-1)/x)f_\mathrm{o}(x)$. For the $p = 1$ case, D reduces to the ordinary derivative operator d. In the realization of parabose algebra for a single degree of freedom the pair (x, D) plays the same role as (x, d) in the case of realizations of the ordinary boson algebras. For example, the parabose algebra for a single degree of freedom

$$[a, \{a^\dagger, a\}] = 2a \qquad [a, \{a^\dagger, a^\dagger\}] = 4a^\dagger \qquad [a, \{a, a\}] = 0 \tag{12}$$

has the familiar realization

$$a = \frac{x + D}{\sqrt{2}} \qquad a^\dagger = \frac{x - D}{\sqrt{2}} \qquad (P = -\mathrm{i}D). \tag{13}$$

† Notice that equation (10) is different from Kamefuchi's and the difference is crucial for defining the deformed calculus described in the next section.

Like the q-deformed calculus in which the q-analogue of the number system is defined by [13] $[n]_q = ((q^n - 1)/(q - 1))$, such that when $q \to 1$, $[n]_q \to n$, in our present case, we can introduce a new kind of deformed number system which is defined by

$$[n] = n + \frac{p-1}{2}(1 - (-1)^n). \tag{14}$$

Obviously, $[2k] = 2k$, $[2k+1] = 2k + p$ for any integer k, and when $p \to 1$, $[n] \to n$. The paraquatization order p is referred to as a deformation parameter in the present case. Generalization of the ordinary differential equation $\mathrm{d}x^n = nx^{n-1}$ reads

$$Dx^n = [n]x^{n-1} \tag{15}$$

which reveals the effect of the deformed derivative operator D defined by equation (11) on the polynomials of x. If we introduce a notation $E(x)$ defined by

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \qquad [n]! = [n][n-1]\ldots[1] \qquad [0]! \equiv 1. \tag{16}$$

We also have

$$DE(x) = E(x). \tag{17}$$

Therefore, $E(x)$ is a deformation of the ordinary exponential function e^x in our case and it will reduce to e^x when $p \to 1$. The operator D defined by equation (11) acts on a product of two functions of the real variable x, say $f(x)$ and $g(x)$, and gives

$$D(fg) = (Df)g + Rf\,Dg + (f - Rf)g' = f\,Dg + (Df)Rg + f'(g - Rg) \tag{18}$$

where f' stands for $\mathrm{d}f/\mathrm{d}x$. When either $f(x)$ or $g(x)$ is an even function of x, i.e., $Rf(x) = f(x)$ or $Rg(x) = g(x)$, equation (18) will give the corresponding Leibnitz rule in the present deformed case

$$D(fg) = (Df)g + f\,Dg. \tag{19}$$

Inversion of the deformed derivative formula (11) leads to new deformed integration also. To see this, let us first write

$$Df(x) = \left(1 + \frac{p-1}{2x}(1-R)\,\mathrm{d}^{-1}\right)\mathrm{d}f(x) = F(x) \tag{20}$$

here, the operation d^{-1} is taken to be the usual integration $\int \mathrm{d}x$. Then, the definition of the deformed integration in this case is obtained formally as follows

$$\begin{aligned} f(x) = D^{-1}F(x) &= \mathrm{d}^{-1}\left(1 + \frac{p-1}{2x}(1-R)\,\mathrm{d}^{-1}\right)^{-1} F(x) \\ &= \mathrm{d}^{-1}\left(1 - \frac{p-1}{2x}(1-R)\,\mathrm{d}^{-1} + \frac{p-1}{2x}(1-R)\,\mathrm{d}^{-1}\frac{p-1}{2x}(1-R)\,\mathrm{d}^{-1} - \cdots\right)F(x) \\ &= \int \mathrm{d}x\,F(x) - \int \mathrm{d}x\,\frac{p-1}{2x}(1-R)\int \mathrm{d}x\,F(x) \\ &\quad + \left(\int \mathrm{d}x\,\frac{p-1}{2x}(1-R)\right)^2 \int \mathrm{d}x\,F(x) - \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\int \mathrm{d}x\,\frac{p-1}{2x}(1-R)\right)^n \int \mathrm{d}x\,F(x) \equiv \int Dx\,F(x). \end{aligned} \tag{21}$$

i.e.

$$\int Dx\,F(x) = \int \mathrm{d}x\,F(x) + \sum_{n=1}^{\infty}(-1)^n(p-1)^n\left(\int\frac{\mathrm{d}x}{x}\right)^n\int\frac{\mathrm{d}x}{2}(F(x)+F(-x)). \tag{22}$$

In the limit $p \to 1$, (22) reduces to the usual integration. To check formula (22), let us take $F(x) = x^n$. Substituting it into equation (22), we have

$$\begin{aligned}\int Dx\,x^n &= \int \mathrm{d}x\,x^n - \int \mathrm{d}x\frac{p-1}{2x}(1-R)\int \mathrm{d}x\,x^n + \left(\int \mathrm{d}x\frac{p-1}{2x}(1-R)\right)^2\int \mathrm{d}x\,x^n - \cdots \\ &= \frac{x^{n+1}}{n+1} + c - \frac{p-1}{2}\frac{1+(-1)^n}{n+1}\frac{x^{n+1}}{n+1} + \left(\frac{p-1}{2}\frac{1+(-1)^n}{n+1}\right)^2\frac{x^{n+1}}{n+1} - \cdots \\ &= \sum_{k=0}^{\infty}(-1)^k\left(\frac{p-1}{2}\frac{1+(-1)^n}{n+1}\right)^k\frac{x^{n+1}}{n+1} + c = \frac{x^{n+1}}{[n+1]} + c\end{aligned} \tag{23}$$

as it should be, where c is an integration constant. Similarly, we have

$$\int Dx\,E(x) = E(x) + c. \tag{24}$$

From the definition of equation (21), it is easily seen that if $F(x)$ is an odd function of x, $F(-x) = -F(x)$, its deformed integration will reduce to the ordinary integration, that is, $\int Dx\,F(x) = \int \mathrm{d}x\,F(x)$. Equation (21) gives us a formal definition for our deformed integration in the sense of the indefinite integral. For the definite integral, we have

$$\begin{aligned}\int_a^b Dx\,F(x) &\equiv \int_a^b \mathrm{d}x\,F(x) - \int_a^b \mathrm{d}x\frac{p-1}{2x}(1-R)\int_a^x \mathrm{d}x\,F(x) \\ &\quad + \int_a^b \mathrm{d}x\frac{p-1}{2x}(1-R)\int_a^x \mathrm{d}x\frac{p-1}{2x}(1-R)\int_a^x \mathrm{d}x\,F(x) - \cdots \\ &= \int_a^b \mathrm{d}x\sum_{n=0}^{\infty}(-1)^n\left(\frac{p-1}{2x}(1-R)\int_a^x \mathrm{d}x\right)^n F(x).\end{aligned} \tag{25}$$

If either $F(x)$ or $G(x)$ is an even function of x, we also have a formula of integration by parts from equation (19)

$$\int_a^b Dx\frac{DF}{Dx}G = FG\,|_a^b - \int_a^b Dx\,F\frac{DG}{Dx}. \tag{26}$$

4. The eigenequation for free parabosonic system

In section 2 we deduced the expression for the momentum operator P in coordinate representation for a single paraboson case. Considering equation (17), we can easily find that up to a normalization constant the eigenfunction of the operator P in the coordinate representation is

$$P\Phi_k(x) = \mathrm{i}k\Phi_k(x) \qquad \Phi_k(x) \propto E(\mathrm{i}kx). \tag{27}$$

Next, let us solve the eigenequation for the free parabose Hamiltonian $H = 1/2(P^2+x^2)$

$$H\Psi_n(x) = \varepsilon_n\Psi_n(x). \tag{28}$$

Substituting equation (10) into this equation we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{p-1}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{p-1}{2x} - x^2 + 2\varepsilon_n\right)\Psi_n(x) = -\frac{p-1}{2x}\Psi_n(-x). \tag{29}$$

As in the ordinary case, for $x \to \pm\infty$, the function $\Psi_n(x)$ must have the form $\Psi_n(x) = \exp(-x^2/2)$. Thus, we can seek a solution in the form

$$\Psi_n(x) = v_n(x)\,\mathrm{e}^{-x^2/2}. \tag{30}$$

By substituting it into equation (29), we find that the equation for the function $v_n(x)$ is of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} v_n(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x} v_n(x) + \frac{p-1}{x}\frac{\mathrm{d}}{\mathrm{d}x} v_n(x) + \left(2\varepsilon_n - p - \frac{p-1}{2x^2}\right) v_n(x) = -\frac{p-1}{2x^2} v_n(-x). \tag{31}$$

Again similarly to the ordinary harmonic oscillator case, in order that $\Psi_n(x)$ should be finite for $x \to \pm\infty$, it is necessary that the solutions v_n should be polynomials of finite order in x. Such solutions actually exist for each non-negative integer n. To each such value of n, there corresponds a polynomial of order n

$$v_n(x) = (-1)^n\,\mathrm{e}^{x^2} D^n\,\mathrm{e}^{-x^2}. \tag{32}$$

In order to prove that these $v_n(x)$ are solutions to equation (31), let us give a recursion relation

$$D^{n+1}\,\mathrm{e}^{-x^2} + 2xD^n\,\mathrm{e}^{-x^2} + 2[n]D^{n-1}\,\mathrm{e}^{-x^2} = 0 \tag{33}$$

which can be easily checked by mathematical induction. Thus, substituting $v_n(x)$ given by equation (32) into (31), we get

$$2\varepsilon_n + p - (p-1)(1-(-1)^n) = 2[n+1] \tag{34}$$

which implies that the eigenvalues ε_n are of

$$\varepsilon_n = n + \frac{p}{2}. \tag{35}$$

In fact, the polynomials $v_n(x)$ given by equation (32) may be considered as a deformation of the normal Hermite polynomials and denoted as

$$H_n^{(p)}(x) = v_n(x) = (-1)^n\,\mathrm{e}^{x^2} D^n\,\mathrm{e}^{-x^2}. \tag{36}$$

We can write out the explicit form of the first few polynomials

$$\begin{aligned} &H_0^{(p)}(x) = 1 \qquad H_1^{(p)}(x) = 2x \qquad H_2^{(p)}(x) = 4x^2 - 2p \\ &H_3^{(p)}(x) = 8x^3 - 4(p+2)x \qquad H_4^{(p)}(x) = 16x^4 - 16(p+2)x^2 + 4p(p+2)\ldots. \end{aligned} \tag{37}$$

We also would like to point out that the generating function for the deformed Hermite polynomials is

$$\mathrm{e}^{-t^2} E(2tx) = \sum_{n=0}^{\infty} \frac{t^n}{[n]!} H_n^{(p)}(x) \tag{38}$$

and

$$H_n^{(p)}(x) = [n]! \sum_{k=0}^{[n/2]'} \frac{(-1)^k (2x)^{n-2k}}{k![n-2k]!} = (-1)^n\,\mathrm{e}^{x^2} D^n\,\mathrm{e}^{-x^2} \tag{39}$$

where $[k]'$, in the summation notation $\sum$, stands for the largest integer smaller than or equal to k.

Substituting equation (36) into equation (30), we find the eigenfunctions of equation (28) as

$$\Psi_n(x) = N_n\,\mathrm{e}^{-x^2/2} H_n^{(p)}(x) = N_n(-1)^n\,\mathrm{e}^{x^2/2} D^n\,\mathrm{e}^{-x^2} \tag{40}$$

where N_n are normalization constants. These eigenfunctions have been normalization according to the deformed integration

$$\int_{-\infty}^{\infty} Dx\, \Psi_n(x)\Psi_m(x) = \delta_{n,m}. \tag{41}$$

For example, the normalized constant of the ground-state function $\Psi_0(x) = N_0\,\mathrm{e}^{-x^2/2}$ is given by

$$N_0^{-2} = 2\int_0^{\infty} Dx\, \mathrm{e}^{-x^2}. \tag{42}$$

To determine other normalized constants N_n, we need a formula

$$\int_{-\infty}^{\infty} Dx\, x^{2n}\,\mathrm{e}^{-x^2} = \frac{[1][3]\ldots[2n-1]}{2^n} N_0^{-2} \qquad (n > 0) \tag{43}$$

which can be easily proved. In fact, using the formula of integration by parts equation (26) and noticing that the functions e^{-x^2} and x^{2n} are both even, we have

$$\begin{aligned}\int_{-\infty}^{\infty} Dx\, x^{2n}\,\mathrm{e}^{-x^2} &= \frac{2}{[2n+1]}\int_0^{\infty} Dx \frac{D\,x^{2n+1}}{Dx}\mathrm{e}^{-x^2} \\ &= \frac{2}{[2n+1]}x^{2n+1}\,\mathrm{e}^{-x^2}\Big|_0^{\infty} - \frac{2}{[2n+1]}\int_0^{\infty} Dx\, x^{2n+1}\frac{D}{Dx}\mathrm{e}^{-x^2} \\ &= \frac{2}{[2n+1]}\int_{-\infty}^{\infty} Dx\, x^{2(n+1)}\,\mathrm{e}^{-x^2}\end{aligned} \tag{44}$$

which means that

$$\begin{aligned}\int_{-\infty}^{\infty} Dx\, x^{2n}\,\mathrm{e}^{-x^2} &= \frac{[2n-1]}{2}\int_{-\infty}^{\infty} Dx\, x^{2(n-1)}\,\mathrm{e}^{-x^2} \\ &= \frac{[2n-1][2n-3]}{2^2}\int_{-\infty}^{\infty} Dx\, x^{2(n-2)}\,\mathrm{e}^{-x^2} = \cdots \\ &= \frac{[2n-1][2n-3]\ldots[1]}{2^n}\int_{-\infty}^{\infty} Dx\, \mathrm{e}^{-x^2} \\ &= \frac{[1][3]\ldots[2n-1]}{2^n}N_0^{-2}\end{aligned} \tag{45}$$

so equation (43) is proved. By virtue of this formula, we obtain

$$N_n = \frac{N_0}{\sqrt{2^n[n]!}}. \tag{46}$$

Obviously, when $p = 1$, N_n coincides with $1/\sqrt[4]{\pi}\sqrt{2^n n!}$, the normalized constant of an ordinary harmonic oscillator.

The relationship between the coordinate representation and the number representation for a parabose system with a single degree of freedom is clear. For example, the function $\Psi_n(x)$ obtained earlier is exactly the wavefunction of the n-paraboson state in the coordinate representation. Using the usual realization for the operators $a^\dagger$ and a, as well as equation (13), one can check that the ground-state function $\Psi_0(x)$ satisfies

$$a\Psi_0(x) = \frac{1}{\sqrt{2}}(x+D)\Psi_0(x) = 0 \tag{47}$$

and

$$aa^\dagger\Psi_0(x) = \frac{1}{2}(x+D)(x-D)\Psi_0(x) = p\Psi_0(x). \tag{48}$$

In conclusion, we construct the coordinate representation theory for the parabose system with a single degree of freedom, which fills a gap in the paraquantization theories. In order to carry out practical calculations on this parabose coordinate representation, we develop a new kind of deformed calculus. Using the deformed calculus, we solve the eigenequation of the free parabose Hamiltonian and obtain the eigenvalue and the normalization constant for each eigenfunction, which corresponds exactly to the result in the well known number representation. It is interesting to generalize the results obtained in this paper to a case with more than one degree of freedom, and work on this is in progress.

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